Pergamon

Int. J. Heat Mass Transfer. Vol. 41, No. 1, pp. 25–32, 1998

Printed in Great Britain
0017-9310/98 $19.00+0.00

PII: S0017-9310(97)00115-4

The solution of an inverse heat conduction problem subject to the specification of energies

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(*Received* 22 *April* 1997)

Abstract—In this paper, the heat conduction equation subject to an initial condition and to the specification of the energies on two portions which partition a finite, one-dimensional slab, is investigated. Once this inverse heat conduction problem is shown to be well-posed, a boundary element method (BEM) is developed for finding the solution numerically. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The specification of energy over a certain part of a heat conductor relates physically to the specification of the relative heat content of a portion of the conductor, whilst for diffusion problems this condition is equivalent to the specification of the fluid mass in a portion of the diffusion domain. In the case of a one-dimensional, finite conductor, Cannon [1] showed that when the temperature behaviour at one of the ends of the material is specified in advance, then there exists a unique temperature distribution in the conductor, which produces a specified energy over a given portion of the conductor. The purpose of this paper is to show that the existence and uniqueness result still holds in the weaker assumptions when no boundary condition is prescribed, but instead the energy is specified on two portions which partition the finite slab geometry. Furthermore, a BEM is developed for finding the solution numerically.

2. PROBLEM FORMULATION

Consider the problem of finding the continuous temperature function $T(x, t)$ in a finite homogeneous conductor of non-dimensional length $L = 1$ and non-dimensional thermal diffusivity $\alpha = 1$, satisfying the linear heat equation, namely:

$$\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t) \quad (x,t) \in (0,1)\times(0,\infty) \quad (1)$$

subject to the continuous, initial condition T_0, i.e.

$$T(x,0) = T_0(x) \quad x \in [0,1] \quad (2)$$

and to the specification of the energies $E_0(t)$ and $E_1(t)$ over the time-dependent portions $(0, s(t))$ and $(s(t), 1)$ for $t > 0$, of the conductor, namely

$$E_0(t) = \int_0^{s(t)} T(x,t)\,\mathrm{d}x$$

$$E_1(t) = \int_{s(t)}^1 T(x,t)\,\mathrm{d}x \quad t \in (0,\infty). \quad (3)$$

3. MATHEMATICAL ANALYSIS

The result for the existence and uniqueness of the solution of equations (1)–(3) can be stated as follows.

Theorem 1 (*existence and uniqueness*)
If

(a) $T_0 \in C^0([0,1])$;
(b) $E_0, E_1 \in C^1([0,\infty])$, $E_0(0) = \int_0^{s(0)} T_0(x)\,\mathrm{d}x$, $E_1(0) = \int_{s(0)}^1 T_0(x)\,\mathrm{d}x$;
(c) $s \in C^1([0,\infty])$, $\inf_{t>0} s(t) > 0$, $\sup_{t>0} s(t) < 1$;
(d) $T(0,0) = T_0(0)$, $T(1,0) = T_0(1)$;

then the inverse boundary value problem (1)–(3) possesses a unique continuous solution, $T(x, t)$, which satisfies the continuous Dirichlet boundary conditions

$$T(0,t) = f_0(t) \quad T(1,t) = f_1(t) \quad t \in [0,\infty) \quad (4)$$

where f_0 and f_1 are unknown continuous functions.

Proof. It will be shown that theorem 1 reduces to solving a pair of coupled Volterra integral equations of the second kind for f_0 and f_1, which degenerate into a single equation, which was considered by Cannon [1] when $f_1(t)$ is assumed known and identically taken to be zero.

From the linearity of the partial differential heat equation (1), it suffices to consider the homogeneous initial condition case, i.e. $T_0(x) \equiv 0$. Then, formally, in terms of the Dirichlet boundary conditions (4), it is well known [2], that the solution of the problem (1), (2) and (4) is given by

$$T(x,t) = -2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)f_0(\tau)\,d\tau + 2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)f_1(\tau)\,d\tau \quad (5)$$

where

$$\theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m,t)$$

$$K(x,t) = \frac{H(t)}{(4\pi t)^{1/2}}\exp\left(-\frac{x^2}{4t}\right) \quad (6)$$

where θ is called the theta function which is positive, continuous with all its partial derivatives continuous, K is the fundamental solution for the linear heat equation and H is the Heaviside function. Using the specification of the energies as given by equation (3), Fubini's theorem and that θ is an even function in the space coordinate, we obtain

$$\frac{E_0(t)}{2} = \frac{1}{2}\int_0^{s(t)} T(x,t)\,dx = \int_0^t \{\theta(0,t-\tau)-\theta(s(t),t-\tau)\}f_0(\tau)\,d\tau + \int_0^t \{\theta(1-s(t),t-\tau)-\theta(1,t-\tau)\}f_1(\tau)\,d\tau \quad (7)$$

$$\frac{E_1(t)}{2} = \frac{1}{2}\int_{s(t)}^1 T(x,t)\,dx = \int_0^t \{\theta(s(t),t-\tau)-\theta(1,t-\tau)\}f_0(\tau)\,d\tau + \int_0^t \{\theta(0,t-\tau)-\theta(1-s(t),t-\tau)\}f_1(\tau)\,d\tau. \quad (8)$$

Adding and subtracting equations (7) and (8) yields

$$\frac{E_0(t)+E_1(t)}{2} = \int_0^t \{\theta(0,t-\tau) - \theta(1,t-\tau)\}(f_0(\tau)+f_1(\tau))\,d\tau \quad (9)$$

$$\frac{E_0(t)-E_1(t)}{2} = \int_0^t \{\theta(0,t-\tau)+\theta(1,t-\tau) - 2\theta(s(t),t-\tau)\}f_0(\tau)\,d\tau - \int_0^t \{\theta(0,t-\tau)+\theta(1,t-\tau) - 2\theta(1-s(t),t-\tau)\}f_1(\tau)\,d\tau \quad (10)$$

which are coupled integral equations to be satisfied by the boundary temperature (4) of any solution of equations (1) and (2).

At this stage, for the simplicity of the presentation, we consider the case $s(t)\equiv 1/2$, although the analysis applies for any time-dependent interface $s(t)$ satisfying condition (c). In this case, equation (10) simplifies to be

$$\frac{E_0(t)-E_1(t)}{2} = \int_0^t \left\{\theta(0,t-\tau)+\theta(1,t-\tau) - 2\theta\left(\frac{1}{2},t-\tau\right)\right\}(f_0(\tau)-f_1(\tau))\,d\tau. \quad (11)$$

Clearly, since in the direct heat conduction problem the continuous solution of equation (1) is determined uniquely by boundary and initial data, there exists a unique continuous solution $T(x,t)$ of the problem given by equations (1), (2) and (4) if, and only if, there exists a unique continuous solution $(f_0(t),f_1(t))$ of equations (9) and (11) satisfying $f_0(0)=f_1(0)=0$. Therefore, the existence and uniqueness of the solution of problem (9) and (11) is now investigated.

Using equation (6) it can readily be seen that equations (9) and (11) can be rewritten as

$$\int_0^t \frac{[f_0(\tau)+f_1(\tau)]}{(4\pi(t-\tau))^{1/2}}\,d\tau = \frac{E_0(t)+E_1(t)}{2} + \int_0^t \theta_0(t-\tau)(f_0(\tau)+f_1(\tau))\,d\tau \quad (12)$$

$$\int_0^t \frac{[f_0(\tau)-f_1(\tau)]}{(4\pi(t-\tau))^{1/2}}\,d\tau = \frac{E_0(t)-E_1(t)}{2} + \int_0^t \theta_1(t-\tau)(f_0(\tau)-f_1(\tau))\,d\tau \quad (13)$$

where

$$\theta_0(t-\tau) = K(1,t-\tau) + \sum_{m=1}^{\infty}\{-2K(2m,t-\tau) + K(2m+1,t-\tau)+K(1-2m,t-\tau)\} \quad (14)$$

$$\theta_1(t-\tau) = 2K\left(\frac{1}{2},t-\tau\right) - K(1,t-\tau) + \sum_{m=1}^{\infty}\left\{-K(2m+1,t-\tau)-K(1-2m,t-\tau) - 2K(2m,t-\tau)+2K\left(\frac{1}{2}+2m,t-\tau\right) + 2K\left(\frac{1}{2}-2m,t-\tau\right)\right\}. \quad (15)$$

Now the hypotheses (b)–(d) enable the inversion theorem for Abel integral equations [3], to be applicable to equations (12) and (13), and, following similarly the arguments from Cannon [1], to yield an equivalent problem in terms of two uncoupled Volterra integral equations of the second kind, namely:

$$f_0(z)+f_1(z) = Q_0(z)+\int_0^z (f_0(\tau)+f_1(\tau))K_0(z,\tau)\,d\tau \quad (16)$$

$$f_0(z)-f_1(z) = Q_1(z)+\int_0^z (f_0(\tau)-f_1(\tau))K_1(z,\tau)\,d\tau \quad (17)$$

where

$$Q_0(z) = \frac{1}{2\pi^{1/2}}\int_0^z \frac{E_0'(t)+E_1'(t)}{(z-t)^{1/2}}\,dt$$

$$Q_1(z) = \frac{1}{2\pi^{1/2}}\int_0^z \frac{E_0'(t)-E_1'(t)}{(z-t)^{1/2}}\,dt \quad (18)$$

$$K_i(z,\tau) = \frac{1}{\pi^{1/2}}\int_\tau^z (z-t)^{-1/2}\frac{d\theta_i}{dt}(t-\tau)\,dt \quad i\in\{0,1\}. \quad (19)$$

We shall prove now that equations (18) and (19) are well-defined, i.e. the integrals involved are convergent, and derive some properties for the functions Q_i and K_i for $i\in\{0,1\}$.

From equation (18), and using hypothesis (b), we obtain

$$|Q_i(z)| \leqslant \frac{2M(z)z^{1/2}}{\pi^{1/2}} \quad i\in\{0,1\} \quad (20)$$

where

$$M(z) = \max\{\max_{t\in[0,z]}|E_0'(t)|, \max_{t\in[0,z]}|E_1'(t)|\}. \quad (21)$$

Hence, Q_i are continuous functions and $Q_i(0) = 0$ for $i\in\{0,1\}$. Integrating by parts equation (19), we obtain

$$K_i(z,\tau) = -\frac{2(z-t)^{1/2}}{\pi^{1/2}}\frac{d\theta_i}{dt}(t-\tau)\Big|_{t=\tau}^{t=z} + \frac{2}{\pi^{1/2}}\int_\tau^z (z-t)^{1/2}\frac{d^2\theta_i}{dt^2}(t-\tau)\,dt \quad i\in\{0,1\}. \quad (22)$$

In order to analyze the well-defineness of K_i in equation (22), from equations (14) and (15) in which we remark that the first coordinate of summation of the kernels K is never zero, it is sufficient to consider the following generic expression

$$S(z,\tau) = -(z-t)^{1/2}\frac{\partial K}{\partial t}(x,t-\tau)\Big|_{t=\tau}^{t=z} + \int_\tau^z (z-t)^{1/2}\frac{\partial^2 K}{\partial t^2}(x,t-\tau)\,dt \quad (23)$$

where x is a non-zero quantity. Using equation (6), and after some calculus, equation (23) may be recast in the form

$$S(z,\tau) = \frac{1}{8\pi^{1/2}}\int_\tau^z (z-t)^{1/2}\exp\left(-\frac{x^2}{4(t-\tau)}\right) \times\left[\frac{3}{(t-\tau)^{5/2}}-\frac{3x^2}{(t-\tau)^{7/2}}+\frac{x^4}{(t-\tau)^{9/2}}\right]dt. \quad (24)$$

Now, when $x\neq 0$, due to the presence of the exponential in equation (24) which decays to zero more rapidly than the powers of $(t-\tau)$ as $t\searrow\tau$, it can be seen that the integrand in equation (24) is not singular provided that $z>\tau$. Therefore, based on the analysis performed in equations (20)–(24), we have obtained that Q_i are continuous functions with $Q_i(0) = 0$ and $K_i(z,\tau)$ are bounded by polynomials in $(z-\tau)^{1/2}$ and are two-dimensionally continuous in both variables for $z>\tau$. Having these properties then [1], the uncoupled Volterra integral equations of the second kind (16) and (17) possess the unique continuous solutions (f_0+f_1) and (f_0-f_1), respectively, such that $(f_0+f_1)(0) = (f_0-f_1)(0) = 0$. Then, the individual boundary temperature functions f_0 and f_1 exist and they are unique satisfying $f_0(0) = f_1(0) = 0$. Furthermore, the stability of the solution is ensured from hypothesis (b) and from the theory of Volterra's integral equations of the second kind as given by equations (16) and (17). Hence, so far we have shown that the inverse problem (1)–(3) is well-posed. The analysis performed in the proof of theorem 1 enables us to state the following representation theorem.

Theorem 2 (*representation theorem*)

If the functions T_0, E_0, E_1 and s satisfy conditions (a)–(d) from theorem 1, then the solution of the inverse boundary value problem (1)–(3) has the representation

$$T(x,t) = \int_0^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]T_0(\xi)\,d\xi - 2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)\phi_0(\tau)\,d\tau + 2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)\phi_1(\tau)\,d\tau \quad (25)$$

if, and only if, ϕ_0 and ϕ_1 are continuous functions that satisfy

$$E_0(t)+E_1(t) - \int_0^1\left\{\int_0^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]T_0(\xi)\,d\xi\right\}dx = 2\int_0^t [\theta(0,t-\tau)-\theta(1,t-\tau)][\phi_0(\tau)+\phi_1(\tau)]\,d\tau \quad (26)$$

$$E_0(t) - E_1(t) - \int_0^1 T_0(\xi)\left\{\int_0^{s(t)} [\theta(x-\xi,t)-\theta(x+\xi,t)]\,\mathrm{d}x - \int_{s(t)}^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]\,\mathrm{d}x\right\}\mathrm{d}\xi$$
$$= 2\int_0^t [\theta(0,t-\tau)+\theta(1,t-\tau)][\phi_0(\tau)-\phi_1(\tau)]\,\mathrm{d}\tau - 4\int_0^t \theta(s(t),t-\tau)\phi_0(\tau)\,\mathrm{d}\tau + 4\int_0^t \theta(s(t)-1,t-\tau)\phi_1(\tau)\,\mathrm{d}\tau. \quad (27)$$

Proof. It is well known [2] that the solution of the direct problem given by equations (1), (2) and (4) is given by

$$T(x,t) = \int_0^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]T_0(\xi)\,\mathrm{d}\xi - 2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)f_0(\tau)\,\mathrm{d}\tau + 2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)f_1(\tau)\,\mathrm{d}\tau. \quad (28)$$

Integrating equation (28) and using the specification of the energies, as given by equation (3), yield

$$E_0(t) - \int_0^{s(t)}\left\{\int_0^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]T_0(\xi)\,\mathrm{d}\xi\right\}\mathrm{d}x = 2\int_0^t [\theta(0,t-\tau)-\theta(s(t),t-\tau)]f_0(\tau)\,\mathrm{d}\tau + 2\int_0^t [\theta(s(t)-1,t-\tau)-\theta(-1,t-\tau)]f_1(\tau)\,\mathrm{d}\tau \quad (29)$$

$$E_1(t) - \int_{s(t)}^1\left\{\int_0^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]T_0(\xi)\,\mathrm{d}\xi\right\}\mathrm{d}x = 2\int_0^t [\theta(s(t),t-\tau)-\theta(1,t-\tau)]f_0(\tau)\,\mathrm{d}\tau + 2\int_0^t [\theta(0,t-\tau)-\theta(s(t)-1,t-\tau)]f_1(\tau)\,\mathrm{d}\tau. \quad (30)$$

Adding and subtracting equations (29) and (30) yields

$$E_0(t)+E_1(t) - \int_0^1\left\{\int_0^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]T_0(\xi)\,\mathrm{d}\xi\right\}\mathrm{d}x = 2\int_0^t [\theta(0,t-\tau)-\theta(1,t-\tau)][f_0(\tau)+f_1(\tau)]\,\mathrm{d}\tau \quad (31)$$

$$E_0(t) - E_1(t) - \int_0^1 T_0(\xi)\left\{\int_0^{s(t)} [\theta(x-\xi,t)-\theta(x+\xi,t)]\,\mathrm{d}x - \int_{s(t)}^1 [\theta(x-\xi,t)-\theta(x+\xi,t)]\,\mathrm{d}x\right\}\mathrm{d}\xi$$
$$= 2\int_0^t [\theta(0,t-\tau)+\theta(1,t-\tau)][f_0(\tau)-f_1(\tau)]\,\mathrm{d}\tau - 4\int_0^t \theta(s(t),t-\tau)f_0(\tau)\,\mathrm{d}\tau + 4\int_0^t \theta(s(t)-1,t-\tau)f_1(\tau)\,\mathrm{d}\tau. \quad (32)$$

Now the representation theorem stated in equations (25)–(27) follows immediately by replacing f_0 and f_1 in equations (28), (31) and (32) by ϕ_0 and ϕ_1, respectively.

At this stage, it should be noted that, although theoretically interesting, the representation theorem 2 does not give explicitly the solution of the inverse problem (1)–(3). Therefore, the solution of the problem (1)–(3) can actually be obtained only numerically, as described in the next section.

4. NUMERICAL ANALYSIS

Initially it should be noted that an attempt to find the numerical solution of the two uncoupled Volterra integral equations of the second kind (16) and (17) will be very cumbersome because of the complicated nature of the kernels K_i given by equation (19), via equations (6), (14) and (15), involving infinite series evaluations, and also because of the expressions of the free terms Q_i given by equation (18) involving derivatives of the energies $E_i(t)$. In addition, it is worth noting that, in practice, seldom will cases measure smooth functions for the energy functions, as assumed in the hypothesis (b) of theorem 1. The most we can hope for from a practical measurement is a continuous function which usually is not smooth. In such a situation, the continuous data $E_i(t)$ can be mollified, i.e. filtered, into a more restricted class of functions, say C^s, $s \geqslant 1$, e.g. cubic splines, and then an approximate problem can be solved within the hypotheses (a)–(d) of theorem 1. In order to overcome these difficulties, the numerical method employed in this study for solving the problem (1)–(3) is the BEM. Since the governing heat conduction equation (1) is linear it is recognized [4] that in such a situation the BEM is the best numerical discretisation method.

It is well known [5] that the solution of equations (1), (2) and (4) can be reformulated in an integral form as

$$\eta(x)T(x,t) = \int_0^t G(x,t;0,\tau)T'(0,\tau)\,d\tau + \int_0^t G(x,t;1,\tau)T'(1,\tau)\,d\tau - \int_0^t G'(x,t;0,\tau)T(0,\tau)\,d\tau - \int_0^t G'(x,t;1,\tau)T(1,\tau)\,d\tau + \int_0^1 G(x,t;y,0)T_0(y)\,dy \quad (33)$$

where $\eta(x) = 0.5$ if $x \in \{0, 1\}$ and $\eta(x) = 1$ if $x \in (0, 1)$. Primes denote the differentiation with respect to the outward normal at the boundaries $x = 0$ and $x = L = 1$ of the finite slab $[0, L]$ and

$$G(x,t;\xi,\tau) = K(x-\xi, t-\tau). \quad (34)$$

As the specification of the energies given by equations (3) is yet to be utilized, we integrate equation (33) to yield

$$E_0(t) = \int_0^{s(t)} \frac{1}{\eta(x)}\left[\int_0^t (G(x,t;0,\tau)T'(0,\tau) + G(x,t;1,\tau)T'(1,\tau))\,d\tau\right]dx - \int_0^{s(t)} \frac{1}{\eta(x)}\left[\int_0^t (G'(x,t;0,\tau)T(0,\tau) + G'(x,t;1,\tau)T(1,\tau))\,d\tau\right]dx + \int_0^{s(t)} \frac{1}{\eta(x)}\int_0^1 G(x,t;y,0)T_0(y)\,dy\,dx \quad (35)$$

and a similar expression exists for $E_1(t)$.

Since an analytical solution of the integral equation (35) is basically impossible then a numerical approximation is necessary. For simplicity, in the formulation of the BEM applied in this section, constant time elements, i.e. the boundary temperature and heat flux are assumed constant over each time element, are used when the integral equation (35) is discretized. The discretization of equation (35) is global, i.e. non-marching in time, and is such that a time interval of interest, say $[0, t_f]$, is divided into N equidistant time elements on each boundary $x = 0$ and $x = L = 1$, whilst the space interval $[0, L]$ is divided into N_0 cell elements. With this assumption, the approximation of the equation (35) results in

$$E_0(\tilde{t}_i) = \int_0^{s(\tilde{t}_i)} \frac{1}{\eta(x)}\left\{\sum_{j=1}^{i}\left[T'_{0j}\int_{t_{j-1}}^{t_j} G(x,\tilde{t}_i;0,\tau)\,d\tau + T'_{1j}\int_{t_{j-1}}^{t_j} G(x,\tilde{t}_i;1,\tau)\,d\tau\right]\right\}dx - \int_0^{s(\tilde{t}_i)} \frac{1}{\eta(x)}\left\{\sum_{j=1}^{i}\left[T_{0j}\int_{t_{j-1}}^{t_j} G'(x,\tilde{t}_i;0,\tau)\,d\tau + T_{1j}\int_{t_{j-1}}^{t_j} G'(x,\tilde{t}_i;1,\tau)\,d\tau\right]\right\}dx + \int_0^{s(\tilde{t}_i)} \frac{1}{\eta(x)}\left[\sum_{k=1}^{N_0} T_0^k \int_{y_{k-1}}^{y_k} G(x,\tilde{t}_i;y,0)\,dy\right]dx$$

$$i = \overline{1,N} \quad (36)$$

and a similar expression for $E_1(\tilde{t}_i)$, where t_{j-1} and t_j are the endpoints of a time element, $\tilde{t}_i = (t_{j-1}+t_j)/2$, y_{k-1} and y_k are the endpoints of a cell element, $\tilde{y}_k = (y_{k-1}+y_k)/2$, $T_{0j} = T(0,\tilde{t}_j)$, $T_{1j} = T(1,\tilde{t}_j)$, $T'_{0j} = T'(0,\tilde{t}_j)$, $T'_{1j} = T'(1,\tilde{t}_j)$ and $T_0^k = T_0(\tilde{y}_k)$.

If for $i, j = \overline{1,N}$, $k = \overline{1,N_0}$ and $l \in \{0, 1\}$, we denote

$$X_{ilj} = \int_0^{s(\tilde{t}_i)} \frac{1}{\eta(x)}\left[\int_{t_{j-1}}^{t_j} G(x,\tilde{t}_i;l,\tau)\,d\tau\right]dx \quad (37)$$

$$Y_{ilj} = -\int_0^{s(\tilde{t}_i)} \frac{1}{\eta(x)}\left[\int_{t_{j-1}}^{t_j} G'(x,\tilde{t}_i;l,\tau)\,d\tau\right]dx \quad (38)$$

$$Z_{ik} = \int_0^{s(\tilde{t}_i)} \frac{1}{\eta(x)}\left[\int_{y_{k-1}}^{y_k} G(x,\tilde{t}_i;y,0)\,dy\right]dx \quad (39)$$

$$A = (A_{ij}) = \begin{cases} Y_{i0j} \\ Y_{i1j} \\ X_{i0j} \\ X_{i1j} \end{cases} \quad \mathbf{b} = (b_i) = E_0(\tilde{t}_i) - \sum_{k=1}^{N_0} Z_{ik}T_0^k \quad (40)$$

$$\mathbf{x} = \begin{cases} T_{0j} \\ T_{1j} \\ T'_{0j} \\ T'_{1j} \end{cases} \quad (41)$$

then equation (36) can be rewritten as a system of linear equations, namely:

$$A\mathbf{x} = \mathbf{b}. \quad (42)$$

The number of unknowns in equation (42) is the dimension of the vector $\mathbf{x}$, i.e. $4N$, see equation (41). Clearly from equation (40), equation (42) provides us with N equations. Another N equations are provided in a similar way from the expression for $E_1(t)$ analogous to equations (35) and (36), only that the limits of integration in these equations are from $s(t)$ to 1 instead of from 0 to $s(t)$. The remaining $2N$ equations are obtained by taking x on the boundary, i.e. $x \in \{0, 1\}$, in equation (33). The latter derivation of the

$2N$ equations are common in the boundary element methodology and, therefore, the details are omitted herein.

Hence, at the end of the numerical analysis performed in this section, the problem of finding the vector of unknowns **x** defined by equation (41), reduces to solving a system of $4N$ linear equations with $4N$ unknowns. Since we have shown in Section 3 that the inverse problem (1)–(3) is well-posed, this system of equations will be well-conditioned and, therefore, a direct method, such as a Gaussian elimination procedure, can be used.

Finally, we note that the integrals with respect to τ and y in equations (37)–(39) can be evaluated analytically [6], whilst the integrals with respect to x can be evaluated numerically using a midpoint rectangular or trapezoidal rule. In the next section the numerical results are compared with the exact solution of a simple benchmark test example.

5. NUMERICAL RESULTS AND DISCUSSION

In order to show the convergence and accuracy of the BEM developed in the previous section, we choose a simple benchmark test example, namely:

$$s(t) \equiv 0.5 \quad T(x,t) = x^2 + 2t \quad (x,t) \in (0,1) \times (0,1). \tag{43}$$

In this case, the initial and energies specification data given by equations (2) and (3) result immediately in

$$T_0(x) = x^2 \quad E_0(t) = t + \tfrac{1}{24} \quad E_1(t) = t + \tfrac{7}{24}. \tag{44}$$

The desired boundary data defined by equation (4) is then given by

$$f_0(t) = 2t \quad f_1(t) = 1 + 2t. \tag{45}$$

The BEM numerical results are compared with the exact Dirichlet boundary data (45) in Fig. 1 for various numbers of time elements $N \in \{10, 20, 40\}$ and for a fixed number of cells $N_0 = 20$.

From Fig. 1 it can be seen that the numerical results for $f_0(t)$ and $f_1(t)$ are in very good agreement with the exact values (45) even when a relative coarse mesh size, i.e. $N = 10$ boundary elements, is used for discretization.

It should be noted that, unlike other numerical methods, the BEM simultaneously provides the boundary temperature and the heat flux. The BEM numerical results for the heat fluxes are compared with their exact values, namely:

$$q_0(t) = -\frac{\partial T}{\partial x}(0,t) = 0 \quad q_1(t) = \frac{\partial T}{\partial x}(1,t) = 2 \tag{46}$$

in Fig. 2 for various numbers of time elements $N \in \{10, 20, 40\}$ and for a fixed number of cells $N_0 = 20$.

From Fig. 2 it can be seen that, as expected, the heat flux is more difficult to calculate accurately than the boundary temperature. Slightly larger errors are obtained near $t = 0$ due to the presence of the corners $(x,t) \in \{(0,0),(1,0)\}$, which slow down the rate of convergence of any numerical method that one may

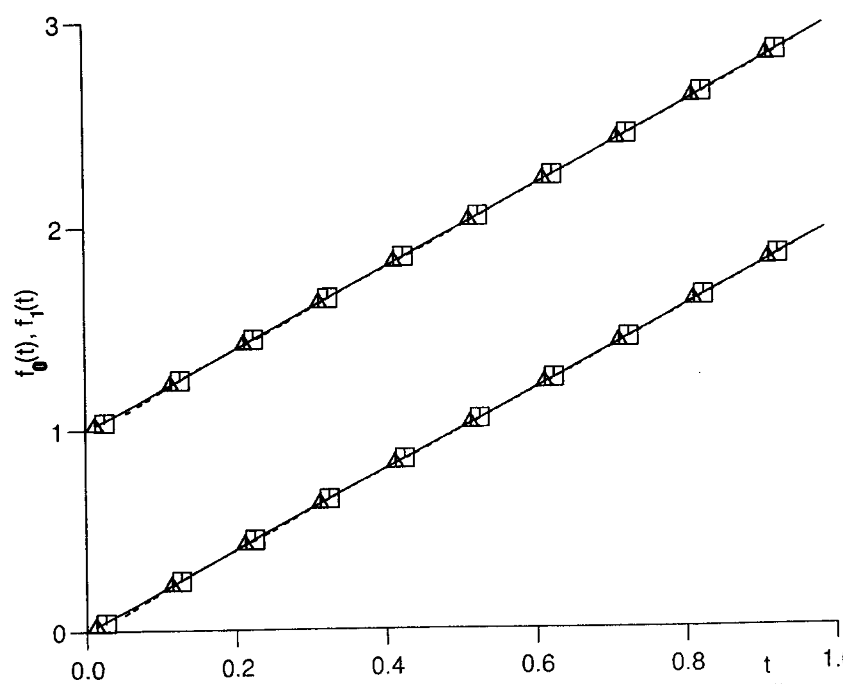

Fig. 1. The BEM numerical results for $f_0(t)$ and $f_1(t)$, obtained for a fixed number of cells $N_0 = 20$ and various numbers of time elements, namely, $N = 10$ (– – –), $N = 20$ (–□–), $N = 40$ (–△–) and the exact solution (——) given by equation (45).

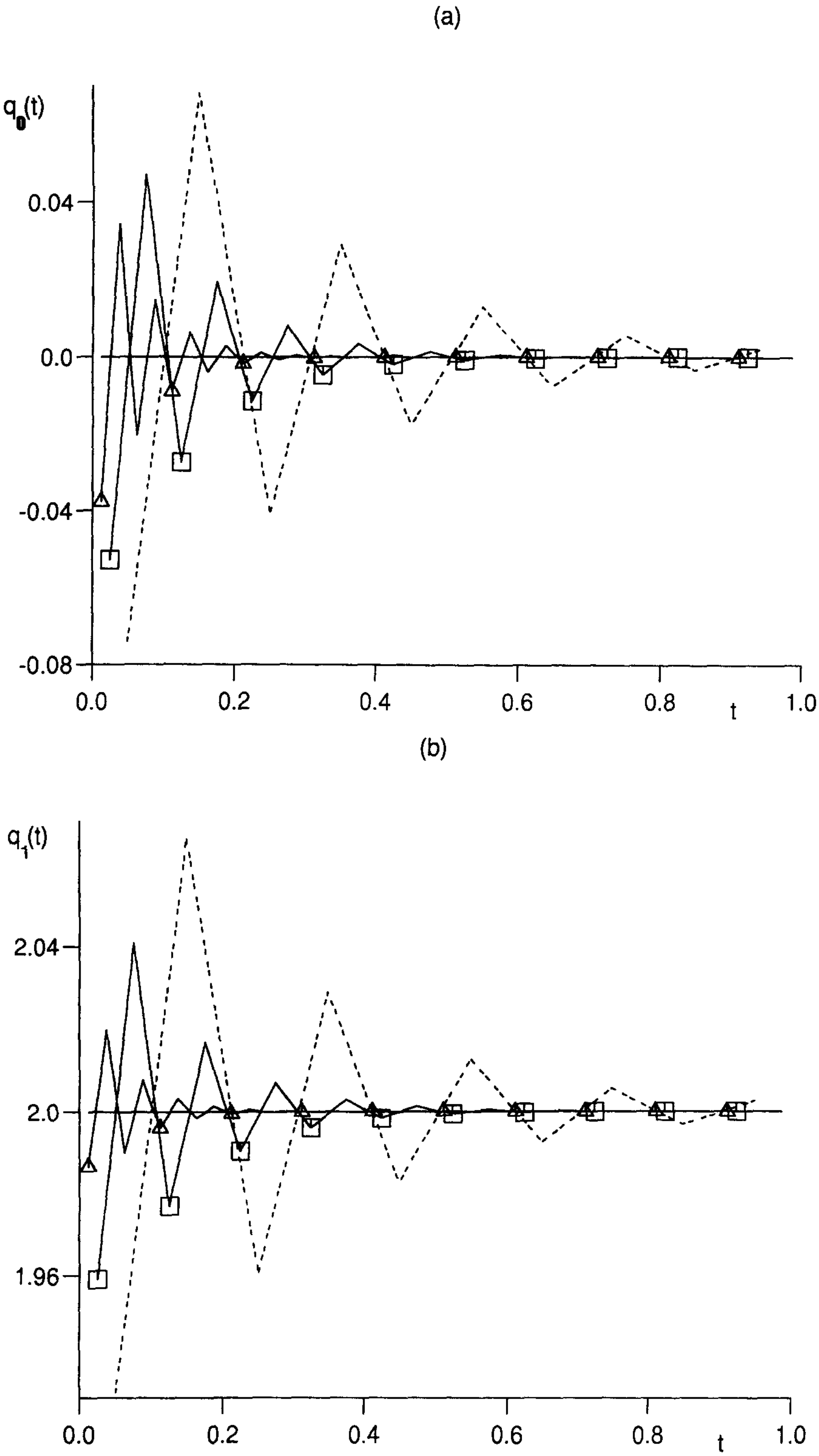

Fig. 2. The BEM numerical results for (a) $q_0(t)$ and (b) $q_1(t)$, obtained for a fixed number of cells $N_0 = 20$ and various numbers of time elements, namely, $N = 10$ (– – –), $N = 20$ (-□-), $N = 40$ (-△-) and the exact solution (——) given by equation (46).

employ. However, if the mesh size is refined, in this case from $N = 10$ to $N = 40$, the convergence of the numerical results for $q_0(t)$ and $q_1(t)$ towards the exact values (46) is achieved.

Once the boundary temperature and heat flux have been determined, the temperature distribution, $T(x, t)$, inside the domain can be obtained explicitly from equation (33). Although not graphically illustrated, it is reported that the lines of constant temperature obtained numerically using the finer mesh size $N = 40$ and $N_0 = 20$, were indistinguishable from those obtained using the exact solution given by equation (43).

Overall from Figs 1 and 2 it can be concluded that the BEM developed and tested in Sections 4 and 5, produces a stable, convergent and accurate numerical solution for the inverse well-posed problem (1)–(3).

6. CONCLUSIONS

The purpose of this paper was to extend the result of Cannon [1] for solving the inverse heat conduction problem when no boundary condition is prescribed, but instead the energy is specified on two portions which partition a finite slab heat conductor. In such a formulation, the inverse problem reduces to solving two Volterra integral equations of the second kind, see equations (16) and (17), for which the existence and uniqueness of the boundary temperature have been established. Furthermore, a BEM has been developed for finding the solution numerically. Finally, it should be noted that the results presented in this paper also hold if, instead of specifying the energies as given by equation (3), we prescribe the differential temperatures, namely:

$$D_0(t) = \int_0^{s(t)} \frac{\partial T}{\partial x}(x,t)\,\mathrm{d}x = T(s(t),t) - T(0,t) \quad t \in (0,\infty) \qquad (47)$$

$$D_1(t) = \int_{s(t)}^{1} \frac{\partial T}{\partial x}(x,t)\,\mathrm{d}x = T(1,t) - T(s(t),t) \quad t \in (0,\infty) \qquad (48)$$

but with the mathematical analysis in the heat flux, rather than in the boundary temperature as in equation (4).

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